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A Theory of Dynamic Oligopoly, I:

Overview and Quantity Competition with Large Fixed Costs

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A THEORY OF DYNAMIC OLIGOPOLY, I:

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and

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Introduction to the Series

In this series of three papers, we present a theory of how oligopolistic firms behave over time. One of our goals is to study in an explicitly temporal model certain ideas that are implicitly dynamic but heretofore have been formally expressed only statically. Throughout we stress the relationship between two fundamental concepts - commitment and reaction.

When we say that a firm is committed to a particular action, we mean that it cannot change that action for a certain finite (although, possibly short) period, during which time other firms might act. By a firm's reaction to another firm we mean the response it makes, possibly after some lag, to the other's chosen action.

To formalize the ideas of commitment and reaction, we introduce a class of infinite-horizon sequential duopoly games. In all these games, the two firms move alternatingly. Firms maximize their discounted sum of single-period profits, and our goal is to characterize the perfect equilibria. The fact that, once it has moved, a firm cannot move again for two periods implies a degree of commitment. We assume, moreover, that each firm uses a strategy that makes its move in a given period a function only of the other firm's most recent move. Our primary justification for this simplifying assumption - we provide a lengthier defense below - is that it makes strategies dependent only on the physical state of the system, those variables that are directly payoff-relevant. Consequently, we can speak legitimately of a firm's reaction to another's action, rather than to an entire history of actions by both firms.

Of course, the assumption that firms' moves necessarily alternate is artificial; we could just as well suppose that moves are simultaneous.

To provide proper foundations for our alternating move hypothesis, therefore, we consider a more elaborate class of models where firms can, in principle, move at any time they choose. Yet, as before, once a firm selects a move, it remains committed to that action for a certain length of time. When we restrict attention to strategies that are functions only of the physical state, we find that, in many cases of interest, the equilibrium behavior in endogenous timing models closely parallells that in the games where alternation is imposed; that is, firms choose in equilibrium to move in turn.

To place our work in perspective, we begin in Section 2 with a brief, and highly selective, review of the literature on dynamic oligopoly. In Section 3 we turn to a discussion of the general features of our simple alternating move models. Section 4 introduces the further complication of endogenous timing. Since the original contributions of Cournot (1838) and Bertrand (1883), the most popular strategic variables in models of oligopoly have been quantities (levels of output sold) and prices. Indeed, our own work divides between quantity and price games. In Section 5, we offer our views about the relative merits of these two variables and suggest when we think each is appropriate.

Then, in Section 6, we begin the formal analysis of our series with a study of quantity models in which fixed costs are so large that at most one firm can make a profit. For the exogenous timing version of the model, we show that there exists a unique symmetric (perfect) equilibrium. In this equilibrium, only one firm produces and, furthermore, for discount factors that are not too low, operates above the pure monopoly level in order to deter entry. Such behavior can be thought of as the quantity-analogue of limit pricing behavior (see Gaskins (1971), Kamien and Schwartz (1971), and

Pyatt (1971)). Moreover, as the discount factor tends to one, so that future profit becomes increasingly important, the entry-deterring quantity approaches the competitive (i.e., zero-profit level), a result much in the spirit of the recent contestability literature (see, for example, Baumol, Panzar, and Willig (1982)). The results in Section 7 demonstrate that, with certain qualifications, our conclusions of Section 6 still hold in a model with endogenous timing. We make a few concluding remarks in Section 8.

The second and third papers of our series consider other applications of our general dynamic models. Paper II considers quantity models very much like those of Sections 6 and 7, only <u>without</u> large fixed costs. Thus the models are dynamic analogues of the classic Cournot (1838) construct. We are particularly concerned here with the relative timing of firms' move. We show that even if because of short-term rigidities firms could move alternatingly, they nonetheless will end up acting simultaneously in the steady state.

Paper III studies models of price competition in markets with undifferentiated commodities. We show that two classical phenomena, the kinked demand curve equilibrium and the Edgeworth cycle, arise naturally as equilibria of our models. We also examine the use of market shares and excess capacity as strategic variables.

2. A Brief Review of the Literature[1]

The early architects of the theory of oligopoly, Cournot (1838) and Bertrand (1883), modelled the competition between firms in a market as a simultaneous choice of actions, quantities for the former author, prices

[1] For a far more comprehensive survey of developments up to the mid-1970's, see Friedman (1977).

for the latter. Historically Cournot's approach has attracted considerably more attention than that of Bertrand. This is perhaps because, in the simplest Bertrand model, with constant (and identical) costs for all firms, one equilibrium (often the only equilibrium) is always perfect competition (price equals marginal cost) as soon as there are at least two firms. Perfect competition is widely regarded as too extreme an outcome in markets with very few firms. In these models, firms act to maximize profits given their predictions of other firms' actions. Although one speaks of a Cournot or Bertrand "reaction function," the locus of a firm's best responses to the actions of others, the term is misleading since, given the simultaneity, there is no genuine opportunity for a firm to genuinely respond. Indeed, in equilibrium, one observes only a single point on the reaction function, namely, the equilibrium price or quantity. Of course, out of equilibrium, other points might be observed. Indeed, there is a considerable literature on various adjustment processes in which at each stage, all firms choose actions to maximize profit in that stage, taking other firms' last periods actions as given (the Cournot tâtonnement). The literature has emphasized the question of stability - whether the adjustment process converges to a steady-state. Its most obvious drawback is the hypothesis it attributes to firms that others will not revise their actions, a hypothesis that is falsified at every step of the process.

The original Cournot and Bertrand models are static, but subsequent theoretical developments recognized the intrinsically dynamic nature of oligopoly. Before the Second World War, three major modifications of the Bertrand-Cournot theorems were introduced.

Stackelberg (1952) assumed temporal asymmetry between firms. In his model of duopoly, one firm, the leader, moves first by selecting a

quantity to which the follower then reacts. There are two main criticisms of this formulation. First it is typically preferable in quantity games to be the leader rather than the follower, but von Stackelberg gave no indication of what determines who leads and who follows. This criticism is especially troubling if firms are innately identical. Second, the limitation to two periods is highly restrictive.

The other two lines involve competition in prices. Edgeworth (1925), in his celebrated criticism of Bertrand, introduced capacity constraints and showed that, in general, no equilibrium in the modified static model exists. He went on to suggest the possibility of price cycles, with a succession of price wars and "truces."

Finally, Hall and Hitch (1939) and Sweezy (1939) proposed a kinked demand curve hypothesis. According to this theory and in contrast with that of Bertrand, prices well above marginal cost can be sustained in an oligopolistic industry by each firm's belief that if it attempts to expand its clientele by lowering its price, other firms will follow suit. In the third paper of our series, we describe Edgeworth cycles and kinked demand curve equilibria more carefully and show how these two phenomena arise quite naturally as equilibria in a fully dynamic model.

By now it is recognized that many of the most interesting aspects of oligopoly are dynamic. Two methods of studying dynamics have been followed. One consists of incorporating intertemporal features (such as reactions) implicitly in the choice of a static solution concept. Work that adopts this method includes the literatures on conjectural variation (see Bowley (1924), Bresnahan (1981), Ulph (1981), and Hahn (1977)), reaction function equilibria (e.g., Spence (1979) and Friedman (1968)), convolutions (see

Marschak and Selten (1974)), and supply function equilibria (see, e.g., Grossman (1981) and Simon (1981)).

There are several fairly obvious drawbacks to this route. First, on a purely game theoretic level, the reduction of a dynamic process to a one-shot game necessarily entails players' having false conjectures. Since, by definition, there is no opportunity to react in a one-shot game, anything other than "Nash-like" beliefs must be literally untrue. Second, several proposed solution concepts typically give rise to a very wide range of equilibria. If the theories are to have much predictive power, there must be some means of selecting among them. But the methods sometimes suggested are rather arbitrary. More seriously, it is often difficult to assess the extent to which a static model accurately reflects the actual dynamic process for which it is a surrogate. Of course, a similar charge can be levelled at the simplifying assumptions of any model. Nonetheless if the dynamics are tractable, it seems preferable to make them explicit.

There have been several attempts to do just that. On the one hand, Levine (1981) has studied models of firms that learn over time and adapt their behavior according rules of thumb. On the other hand there is a rather large literature that presumes full rationality on the part of firms.

Cyert and DeGroot (1970), for example, considered a discrete time, finite horizon, duopoly model in which one firm chooses output levels in even-numbered periods, the other in odd-numbered periods. The payoff in any given period depends on the output levels chosen in that and the previous period. Firms maximize the discounted sum of payoffs. Cyert and DeGroot solve numerically for the perfect equilibrium and estimate the limit of the equilibrium output level when the horizon becomes long.

Friedman (1977), by contrast, used supergames to study oligopoly. A supergame is an infinite repetition of a one-shot game; in each period, the game is played again. In Friedman's model, firms choose quantities simultaneously in every period. Thus the one-shot game is one of imperfect information. A firm's strategy can depend on all previous quantity choices including those of the firm itself. Friedman showed that there is an enormous variety of equilibria in this supergame, a result corresponding to the so-called Folk Theorem of the supergame literature. The equilibria are sustained by the threat of retaliation if any firm ever deviates from the equilibrium path. This idea has been applied to the study of price competition with uncertain demand (Green and Porter (1981)) and with capacity constraints (Brock and Scheinkman (1981)). These latter two contributions work with a restricted set of allowable strategies, called trigger strategies, to reduce the set of equilibria.

3. The General Model with Fixed Timing

We next present the basic features of the simpler (fixed timing) class of models that we analyze in this series.

A. THE MODEL

We consider a duopoly; the model can be generalized to more than two firms but at the expense of simplicity. Each firm i ($i = 1,2$) chooses actions a^i from an action space A. Depending on the interpretation of the model the variable a^i could represent the choice of a price, quantity, location, etc.. It could even represent a vector of choices (see the section on market sharing in Part III of this series for an example of multidimensional actions). Firms act in discrete time, and the horizon is

infinite. Periods are indexed by $t(t=0,1,\ldots)$ and T is the time between two consecutive periods. At time t firm i's instantaneous profit π^i is a function of the current actions of the two firms but not of time:

$$\pi^i = \pi^i(a_t^1, a_t^2)$$

Firms discount future profits with the same interest rate r. Thus their discount factor is

$$\delta = \exp(-rT).$$

Firm i's intertemporal profit can then be written

$$\Pi^i = \sum_{t=0}^{\infty} \delta^t \pi^i(a_t^1, a_t^2).$$

As mentioned earlier, we wish to model the ideas that (a) firms are committed to their actions for a certain length of time, during which time other firms might move, and that (b) they react to the current actions of other firms. The simplest way of accomplishing both objectives is to assume, following Cyert and DeGroot (1970), that firms move sequentially. In odd-numbered periods $(t = 1,3,5,\ldots)$ firm 1 chooses an action to which it is committed for two periods. That is, $a_{2k+2}^1 = a_{2k+1}^1$ for all k. Similarly firm 2 moves in even-numbered periods $(t = 0,2,4,\ldots)$ and $a_{2k+1}^2 = a_{2k}^2$. Thus there is a lag T between a firm's action and its rival's reaction.

We require equilibrium of this model to be perfect. That is, starting from any point in the game tree, the firm to move selects the action that maximizes its intertemporal profit given the subsequent strategies of its rival and itself. As is well-known, perfect equilibrium rules out empty threats, threats that would not be carried out were the circumstances on

which they are contingent to arise. We do not accept any perfect equilibrium, however, but just those that depend only on the "payoff-relevant" history. Specifically, at time $t = 2k$, the only aspect of history that has any "physical" bearing on current or future payoffs is the value of a^1_{2k-1}, for only this variable, among all those before time 2k, enters any instantaneous profit functions from time 2k on. Thus, if the equilibrium is to depend only on payoff-relevant history, firm 2's strategy at time 2k must depend only on a^1_{2k-1}. That is,

$$a^2_{2k} = R^2_{2k}(a^1_{2k-1}).$$

Moreover, because the future appears the same starting from any time period, time itself is not a payoff-relevant variable, and so above we can drop the subscript "2k" from R. Thus, we can represent the firms' behavior - their strategies - by a pair of dynamic reaction functions.

$$R^1 : A \to A$$

and

$$R^2 : A \to A.$$

Actually, although it will not play a major role in this paper, we must allow for the possibility that R^1 and R^2 are random functions, so that $R^1(a^2)$ and $R^2(a^1)$ are, in general, random variables.

Because dynamic reaction functions depend only on the payoff-relevant state of the system, they might alternatively be called "Markov strategies." A pair of reaction functions (R^1, R^2) forms a Markov perfect equilibrium (MPE) if and only if (1) $a = R^2(a^1_{2k-1})$ maximizes firm 2's intertemporal profit at any time 2k, given a^1_{2k-1} and assuming that henceforth each firm i will move according to R^i; and (2) the analogous condition holds for firm

1. Of course, if R^1 and R^2 are random functions we must replace (1) by the statement that each possible relization of $R^2(a^1_{2k-1})$ maximizes firm 2's expected intertemporal profit (we assume risk neutrality). The following proposition is obvious.

Proposition 1: A Markov perfect equilibrium is a perfect equilibrium.

That is, given that its rival ignores all but the pay-off-relevant history, a firm can just as well do the same.

B. THE MARKOV ASSUMPTION

We have several reasons for restricting our attention to Markov strategies. Their most obvious appeal is their simplicity. Firms' strategies depend on as little as possible while still being consistent with rationality.

More relevant from our persepective is that Markov strategies seem to accord much better with the customary conception of a reaction in the informal industrial organization literature than do, say, the reactions generally emphasized in supergames. In supergames, reactions are, typically, threats made to dissuade the rival firm from selecting certain actions. Once an action the threat was meant to discourage is taken, however, there is often no incentive to carry out the threat unless strategies depend not only on one's rival's past actions but one's own as well. This point can be illustrated by reference to the following version of the Prisoners' Dilemma:

	C	D
C	$\frac{17}{8}, \frac{17}{8}$	0, 3
D	3, 0	1, 1

In this game, Player I chooses rows as actions and receives the first number in each box as a payoff. Player II chooses columns as actions and receives the second number as a payoff. Actions are labelled "C" and "D" for "cooperative" and "deviant" respectively. Notice that deviant pair (D,D) constitutes, the only Nash equilibrium if the game is played only once. Suppose instead that the game is repeated infinitely often and that players are interested in the discounted sum of single period payoffs, where δ is the discount factor. As long as $\delta > 7/16$, there exist perfect equilibria of the supergame in which both players play C in every period.

One such equilibrium consists of each player playing C if neither player played D in the previous period, and playing D otherwise. In this equilibrium, player I's threat to play D in response to a deviation by player II is what deters II from deviating. If II has just deviated, however, the only reason I has for carrying out its threat is that II has conditioned its next move on its <u>own</u> past action. One can easily verify that the threat to play D if and only if the other player has just played D cannot be an equilibrium strategy. Moreover one can establish the stronger property that, for $\delta = \frac{1}{2}$, <u>no</u> equilibrium strategies that depend only on the other player's past moves can sustain the cooperative outcome.

The idea that a reacting is following through on a threat is very different from the reasoning behind, say, the kinked-demand curve story. In the kinked-demand curve world cutting one's own price in response to another firm's price cut is not carrying out a threat at all. It is merely an act of self-defense, an attempt to regain lost customers. Put another way, the reaction is a response only to the other firm's price cut and not to earlier history or to reactions of the other firm to itself.

Of course, it is useless to respond (in our sense of the term) to another's action if, by the time one has done so, the other firm has already moved again. That is why our conception of reaction is intimately related to the idea of commitment; it may be worthwhile reacting to a firm's move because the firm is committed to that move, at least for a time.

This point highlights another sense in which our approach differs from the supergame model. In our model there is a physical intertemporal link - one's instantaneous payoff depends not only on current action but on the immediate past. One implication of this intertemporal link is that if we truncate the game so that the horizon is finite rather than infinite, equilibrium in the truncated game depends on the length of the horizon. Moreover, in the models we have studied, if equilibrium converges as the horizon lengthens, it converges to the infinite-period equilibrium. In supergames, by contrast, there is nothing that connects the present with the past or future other than strategies. Hence, all finite truncations have, qualitatively, the same equilibria (at least if the one-shot underlying

game has an unique equilibrium). It is only at infinity that the interesting supergame equilibria appear.

We should add that none of our reasons for restricting to Markov strategies would be persuasive if firms themselves could do better by following more complicated behavior. Yet, such is not the case, for, as we have already pointed out, Markov perfect equilibria are perfect in the usual sense.

C. MARKOV PERFECTION AND DYNAMIC PROGRAMMING

We can solve for a Markov perfect equilibrium by using the game theoretic analogue of dynamic programming. To this end, we define four value functions. Let

$V^1(a^2)$ = the present discounted value of firm 1's profits given that last period firm 2 played a^2 and that henceforth both firms play optimally.

and

$W^1(a^1)$ = The present discounted value of firm 1's profits given that last period firm 1 played a^1 and that henceforth both firms play optimally.

$V^2(a^1)$ and $W^2(a^2)$ are defined symmetrically.

These value functions must be consistent with the dynamic reaction functions. Specifically, we have

$$V^1(a^2) = \max_{a^1} \{\pi^1(a^1,a^2) + \delta W^1(a^1)\}$$

$$= \pi^1(R^1(a^2),a^2) + \delta W^1(R^1(a^2))$$

$$W^1(a^1) = \pi^1(a^1,R^2(a^1)) + \delta V^1(R^2(a^1)),$$

(and analogous equations for V^2 and W^2), where expectation operators should appear before the expressions on the right hand side if R^1 and R^2 are random functions.

4. Endogenous Timing

We admitted in the introduction that the imposition of alternating moves is artificial. There seems no reason why, in principle, firms could not move simultaneously.[1] In this section we extend the alternating move model to allow the relative timing of firms' moves to be endogenously determined. We do so while maintaining the hypothesis that once a firm has moved, it is stuck with that move for a time.

To give firms the opportunity of changing the timing, we introduce the possibility of a "null action." To choose the null action is, in effect, to do nothing, to leave the market. A firm opting for the null action has a zero payoff in that period. It can, however, re-enter the market in any subsequent period. Thus playing the null action enables a firm to bide its time and choose the most propitious moment to act again. To distinguish, however, between a firm that is out of the market and one that is free to move simply because it is no longer committed to its most recent action, we suppose that a firm returning to the market incurs a re-entry cost e.

As in the fixed timing model, instantaneous profit is a function of the current values of the strategic variables, and firms maximize the discounted sum of profits. However it is no longer necessarily true that precisely one firm moves in every period. Depending on what happened in the previous period, both, neither, or only one firm may have the opportunity of moving at a given time. Moreover, although as before, we study Markov perfect equilibria, a Markov strategy is not simply a function of one's rival's last

[1] Of course, literal simultaneity is unlikely. However, that firms act in ignorance of other firms' moves is all that is needed for de facto simultaneity.

move because the payoff relevant state itself is more complicated. In particular, for a firm about to move, it matters (i) whether the other firm moved the previous period, (ii) if so, how, (iii) if not, whether the other firm is currently out of the market, and (iv) whether it itself is currently out of the market. Thus the payoff relevant state on which a Markov strategy depends comprises the answers to these four questions.

Evidently Markov strategies and equilibria are a good deal more complicated than in the basic general model. Nevertheless in two of the three specific cases we consider in this series - the quantity model in Sections 6 and 7 of this paper and the prices models of our third paper - steady-state behavior is essentially the same in both the fixed-timing and extended verions of the model.

5. Prices versus Quantities

Because our study comprises both price and quantity models, we wish to mention our views on the relative appropriateness of each strategic variable. By this point, however, some readers may feel burdened by a surfeit of rather abstract definitions and concepts without illustrative examples. To avoid taxing these readers further with a methodoligical detour, we recommend that they turn directly to the specific fixed-cost quantity models of sections 6 and 7, which amply exemplify our concepts.

Our basic position, apparently shared by Shaked and Sutton (1981)and Kreps and Scheinkman (1982), is that short-run competition is ordinarily conducted through prices and that long-run competition is waged through such instruments as the scale and choice of technology and the choice of product. Adopting quantity as the strategic variable is really a short-hand way of modelling competition through technological scale. That is, when a firm

chooses a quantity level in a Cournot model, it is, in effect, choosing a scale of operation that is appropriate for that level. Left implicit in such a model is the price competition that follows. Thus, when we write "$\bar{\pi}^i = \pi^i(q^1, q^2)$" in a Cournot model, we are really stating that $\bar{\pi}^i$ is the equilibrium profit of firm i in a price game that follows the adoption of scales of operations "corresponding" to output levels q^1 and q^2. The function $\pi^i(q^1, q^2)$ is merely a reduced form.

This point is vividly exemplified by Kreps and Scheinkman (1982). They consider a two-stage model in which, in the first period, duopolisits build capacity at a constant marginal cost c*. Then, in the second period, the firms choose prices simultaneously à la Bertrand-Edgeworth. Each firm supplies the demand it faces at constant marginal cost c** up to capacity. For linear demand and a particular specification of how consumers are rationed if a firm cannot supply its entire demand, Kreps and Scheinkman show that there is a unique perfect equilibrium of the two-stage game. Moreover, it entails each firm's choosing capacity (and production) equal to the equilibrium output level in the classical Cournot model with the same demand function and constant marginal cost c* + c**.

This result is very special in that it depends on the particular rationing scheme and demand function considered. However, to make our point, it does not matter whether equilibrium capacities exactly equal Cournot quantities. We wish only to suggest that, in view of the long-standing discomfort economists have had with the idea that firms set quantities without specifying prices, quantity competition can alternatively be thought of as a surrogate for a more complete model that does incorporate price competition.

6. Quantity Competition with Large Fixed Costs: Fixed Timing

We turn finally to a specific application of our general model, the analysis of markets with large fixed costs. Although we use quantities as the strategic variables, one should regard these as a proxies for choices about the scale of operation, according to the viewpoint expanded in Section 5.

The industrial organization literature has traditionally distinguished among three types of costs of production. Variable costs are incurred only during the period of production and are directly related to the level of output. Fixed costs (measured as a flow) persist only as long as production continues, but are, strictly speaking, independent of scale. Pure sunk costs continue as a liability forever. That is, they are incurred with or without production.

Both fixed and sunk costs have been regarded as barriers to entry. The entry-deterring role of sunk costs is not controversial. When sunk costs take the form of an irreversible investment in nondepreciable capital, a firm's variable cost curves may be forever changed, giving it a permanent advantage over potential entrants or later rivals. This effect has been studied by Spence (1977), (1979), Dixit (1979), and Fudenberg and Tirole (1981). Even when capital is not infinitely durable it may still deter entry as argued by Eaton and Lipsey (1980).

The deterrent that fixed costs create is one of the subjects of the "natural barriers to entry" literature (see Scherer (1980) for a survey). A firm in an oligopolistic industry (one with large fixed costs) can, by virtue of its incumbency, deter entry since the revenue available to a potential entrant does not outweigh the high fixed costs it has to bear. This view has recently been challenged by Grossman (1981) and Baumol, Panzar, and Willig (1982), who maintain that incumbency gives a firm no

privileged position <u>per se</u> if its costs are merely fixed rather than sunk. Such a firm ought not be able to earn substantial positive profit while its potential rivals earn nothing. These authors feel that the threat of entry should drive the profit of the incumbent to zero, the "competitive" level. We shall attempt in this section and the next to reconcile these conflicting views.

Returning to the model of section 3, we shall suppose that two identical firms move alternately and choose nonnegative quantities (more accurately, capacities), q. They maximize the discounted sum of instantaneous profits, with discount factor δ. If q is chosen to be strictly positive, we shall assume that the firm incurs a fixed cost F. Since the firm is committed to the capacity q for two period, we can think of $f = F/1+\delta$ as the per-period equivalent of F. To simplify matters, we assume that variable costs are linear:

variable cost of $q = cq$;

and that demand is also linear:

$$\text{price} = 1 - (q^1 + q^2),$$

where q^i is firm i's value of q. Thus, firm 1's instantaneous profit is

$$(1)\quad \pi^1(q^1, q^2) = \begin{cases} q^1(1-q^1-q^2) - cq^1 - f, & \text{if } q^1 > 0 \\ 0, & \text{if } q^1 = 0, \end{cases}$$

and firm 2's profit is symmetric.

We shall assume that fixed costs are so large that one but not two firms can operate profitably. Specifically, let $\pi^m = d^2/4$, where $d = 1 - c$ (π^m is just monopoly profit gross of fixed costs). Then our profitability assumption requires

$$(2)\quad 2f > \pi^m > f.$$

For comparison, we first consider what these demand and cost assumptions imply about equilibrium in a traditional static Cournot model. In that model, a pair of quantities $(\bar{q}^1,\bar{q}^2)$ is an equilibrium if, for each firm i, $q^i = \bar{q}^i$ maximizes π^i given $\bar{q}^j (j \neq i)$. One can easily verify that, given our demand and cost assumptions, there are three equilibria: $(q^m,0)$, $(0,q^m)$ (where q^m denotes the monopoly level d/2), and a mixed strategy equilibrium in which each firm sets $q = \frac{1-c}{2+d}$ with probability α and with probability $1 - \alpha$ produces nothing, where

$$\alpha = \frac{1-c}{\sqrt{f}} - 2.$$

None of these three equilibria really corresponds to the argument that the threat of entry should drive an incumbent's profit to zero. The two monopolistic equilibria obviously do not; the presence of a second firm has no effect at all. One can maintain that such equilibria are unconvincing because, were the other firm to enter, the incumbent would not keep q at the monopolistic level. But such dynamic considerations are attacks not so much against the <u>equilibria</u> but rather against the static nature of the game itself. There is simply no opportunity in a one-shot, simultaneous move game to react.

The mixed strategy equilibrium perhaps comes closer to capturing the zero-profit story. At least the two identical firms are treated symmetrically and earn zero profits on average. Of course, the equilibrium also has the unfortunate property that, with positive probability, <u>neither</u> firm or <u>both</u> firms operate.

Given the shortcomings of the static quantity model, we turn to an analysis of the equilibrium of our dynamic model. Our main goal is to

exhibit, for each possible value of the discount factor δ, the unique symmetric Markov perfect equilibrium, i.e., the unique equilibrium such that $R^1 = R^2$. Toward this end we first establish a series of characterization lemmas. Throughout we make the cost and demand assumptions (1) and (2).

Lemma 1: Equilibrium dynamic reaction functions R^i are nonincreasing. That is, if $q > q'$ and r and r' are realizations of $R^i(q)$ and $R^i(q')$ respectively, then $r \leq r'$.

Remark: Lemma 1, which does not assume symmetry, is a result that obtains much more generally than in this specific model. The only property of π^i it requires is that the cross partial derivative π^i_{12} be nonpositive and, for $q^i > 0$, strictly negative.

Proof: Suppose that, contrary to our assertion, $q > q'$ but $r > r'$, where, r and r' are realizations of, say, $R^2(q)$ and $R^2(q')$ (recall that the R^i's may be random functions). By definition of R^2, r is a best response to q.

Thus,

(3) $\pi^2(q,r) + \delta W^2(r) \geq \pi^2(q,r') + \delta W^2(r')$.

Similarly,

(4) $\pi^2(q',r') + \delta W^2(r') \geq \pi^2(q',r) + \delta W^2(r)$.

Subtracting (4) from (3), we obtain

$$\pi^2(q,r) - \pi^2(q',r) - \pi^2(q,r') + \pi^2(q',r') \geq 0,$$

which can be rewritten as

(5) $\int_{q'}^{q} \int_{r'}^{r} \pi^2_{12}(x,y)dxdy \geq 0$.

But because $\pi^2_{12}(x,y)$ is nonpositive and, for $y > 0$, strictly negative, inequality (5) is impossible. Q.E.D.

By "leaving the market" we mean choosing $q = 0$. We next show that if firm 1 leaves the market with positive probability in response to a

(positive) move by firm 2 that was, in turn, an optimal reaction to a previous move by firm 1, then firm 1 in fact leaves the market with probability 1.

Lemma 2: In any Markov perfect equilibrium, if 0 is a realization of $R^1(q)$ and $q > 0$ is a realization of $R^2(q')$ for some q', then $R^1(q) = 0$.

Proof: Because reaction functions are non-increasing, $R^1(q+\Delta) = 0$ for any $\Delta > 0$. Thus if $R^1(q) > 0$ with positive probability, $W^2(q+\Delta) > W^2(q)$ for sufficiently small Δ. Thus for sufficiently small Δ, playing q earns firm 2 a strictly lower payoff than $q+\Delta$, a contradiction of the optimality of q.

Q.E.D.

Henceforth we shall confine our attention to symmetric equilibrium (ones where $R^1 = R^2$). We first establish:

Lemma 3: In a symmetric MPE, if r is a positive realization of $R(q)$ (we can drop the superscripts from reaction functions because of symmetry) $r > q$.

Proof: Suppose first that $0 < r < q$. From Lemma 1, $R(r) \geq r$. Moreover, for any realization of r' of $R(r)$, there exists a realization r'' of $R(r')$ such that $r'' \leq R(r)$. Continuing iteratively we find that the firm who responds to q can continue to act optimally in such a way that it always produces no more than the other firm. Thus, in any period where it produces positively, it must lose money - in particular, when it produces r. Since it can ensure itself zero profit by producing nothing, being in the market cannot be optimal. Hence $r < q$ is impossible.

Next suppose that $r = q$. If 0 is a realization of $R(r)$, then from Lemma 2, $R(r) = 0$, an impossibility since $R(r) = R(q)$. Thus all realizations of $R(r)$ must be positive. From the preceding paragraph, $r \leq R(r)$. Thus, repeating the argument of that paragraph, we can once again show that the firm that responds to q can always act optimally in ways that

produces no more than the other firm, which gives us the same contradiction as before. Q.E.D.

We next show that in a symmetric equilibrium, there exists a "deterrence level."

Lemma 4: In a symmetric MPE there exists $\bar{q} > 0$ such that, for all $q > \bar{q}$, $R(q) = 0$, and, for all $q < \bar{q}$, there exists a positive realization of $R(q)$.

Proof: Consider a sequence $\{q_n\}$ tending monotonically to infinity. Suppose that for all n there exists a positive realization r_n of $R(q_n)$. Because of the definition of π^i, $\{r_n\}$ must tend to zero, otherwise the instantaneous payoff becomes unboundedly negative.[1] Hence for sufficiently large n, $q_n > r_n$, a contradiction of Lemma 3. Hence there exists $q' > 0$ such that for all $q > q'$, $R(q) = 0$. Let $\bar{q}$ be the infimum of all such q'. Then for all $q > \bar{q}$, $R(q) = 0$ and, for all $q < \bar{q}$, there exists a positive realization of $R(q)$. Choose $\varepsilon > 0$ so small that $\pi^1(q^m,\varepsilon) > 0$, where q^m is the monopoly quantity. If $\bar{q} = 0$, then $R(\varepsilon) = 0$, and so firm 1 earns zero profit the first period after firm 2 has played ε. Moreover, firm 1 can earn no more than monopoly profit (the theoretical maximum) in any subsequent period. However if firm 1 responds to ε by playing q^m, it earns positive profit the first period, and, if it continues to play q^m, monopoly profit thereafter. Hence $R(\varepsilon) \neq 0$, and so $\bar{q} > 0$.

Q.E.D.

A firm "takes the market" if it chooses q' such that $R(q') = 0$. We next demonstrate that, in response to $q > 0$, a firm either takes or leaves the market.

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[1] This argument may seem to rely on the price becoming negative. However, as long as the marginal cost c is positive, profit goes to negative infinity even if the price is bounded below by zero.

Lemma 5: In a symmetric MPE, for all q and all positive realizations r of $R(q)$, $R(r) = 0$.

Proof: Suppose that, contrary to the Lemma, there exists a positive realization r' of $R(r)$. From Lemma 3, $r > q$ and $r' > r$, a contradiction of Lemma 1. Q.E.D.

We are nearly ready to establish our main proposition, which asserts that, for any $\delta > 0$, there exists a unique symmetric MPE and exhibits that equilibrium explicitly. To state the proposition, we consider the equations

$$(6) \quad \pi(q,q) + \frac{\delta}{1-\delta}\pi(q,0) = 0$$

$$(7) \quad T(q) = \arg\max_{\tilde{q}} \{\pi(\tilde{q},q) + \delta\pi(\tilde{q},0)\}$$

$$(8) \quad \pi(q,q) + \delta\pi(q,0) + \frac{\delta^2}{1-\delta}\pi^m = 0$$

$$(9) \quad \pi(T(q),q) + \delta\pi(T(q),0) + \frac{\delta^2}{1-\delta}\pi^m = 0,$$

where $\pi(x,y) = \pi^1(x,y)$ and π^m is $d^2/4$.

Proposition 2: There exist numbers δ_1, $\delta_2 \in (0,1)$ such that if δ is the firms' discount factor the unique symmetric MPE of the game with instantaneous profit given by (1) and (2) is

$$(10) \quad R(q) = \begin{cases} 0, & q \geq q^* \\ & \\ q^*, & q < q^* \end{cases} \qquad \text{if } \delta_1 \leq \delta < 1$$

$$(11) \quad R(q) = \begin{cases} 0, & q \geq q^{**} \\ q^{**}, & \underline{q} \leq q < q^{**} \\ T(q), & q < \underline{q} \end{cases} \qquad \text{if } \delta_2 \leq \delta < \delta_1$$

and

$$(12)\ R(q) = \begin{cases} 0, & q \geq q^{***} \\ T(q), & q < q^{***} \end{cases} \qquad \text{if } 0 < \delta < \delta_2$$

where q^*, q^{**} and q^{***} are the largest of the roots of (6), (8), and (9) respectively, and $\underline{q}$ solves $T(\underline{q}) = q^{**}$.

Proof: Let $\bar{q}$ be the deterrence level of Lemma 4.

Case I: $\bar{q} \geq q^m$

If $q < \bar{q}$ then there exists a positive realization r of R(q). From Lemma 5 R(r) = 0. Hence from the definition of $\bar{q}$, $r \geq \bar{q}$. We have

$$V(q) = \pi(r,q) + \delta W(r)$$

If $r > \bar{q}$, suppose that a firm responds to q with $(r+\bar{q})/2$ rather than r. Since $\bar{q} > q^m$, $\pi((r+\bar{q})/2,q) > \pi(r,q)$. Furthermore, since $(r+\bar{q})/2 > \bar{q}$, $R((r+\bar{q})/2) = 0$, and so $W((r+\bar{q})/2) > W(r)$. Therefore $(r+\bar{q})/2$ generates higher profit than r, a contradiction. We conclude that for $q < \bar{q}$, the only positive realization of R(q) is $\bar{q}$. Hence, from Lemma 1, $R(q) = \bar{q}$ for all $q < \bar{q}$. From Lemma 5, $R(\bar{q}) = 0$. Therefore, $V(\bar{q}) = 0$.

Now for $q < \bar{q}$,

$$V(q) = \pi(\bar{q},q) + \frac{\delta}{1-\delta}\pi(\bar{q},0).$$

Because $\pi(\bar{q},q)$ is decreasing in q, we have $V(q) > 0$ for all $q < \bar{q}$. Furthermore, for all $q > \bar{q}$ we must have $\pi(\bar{q},q) + \frac{\delta}{1-\delta}\pi(\bar{q},0) < 0$, otherwise $R(q) \neq 0$. Hence $\bar{q}$ must equal q^*, the greatest root of (6). From (6)

$$(13)\ q^* = \frac{d + \sqrt{d^2 - 4(2-\delta)f}}{2(2-\delta)}$$

Thus because $\bar{q} \geq q^m = \frac{d}{2}$, (13) implies

$$(14)\ \delta^2 d^2 - (2d^2 + 4f)\delta + 8f \leq 0.$$

Notice that because $d^2 > 4f$, (14) holds for $\delta=1$. Since it clearly does not hold for $\delta=0$, there exists $\delta_1 \in (0,1)$ such that it holds if and only if $\delta \in [\delta_1, 1]$. Thus $\bar{q} > q^m$ implies that $\delta \in [\delta_1, 1)$ and that (10) holds. Furthermore it is clear that for $\delta_1 \leq \delta < 1$, (10) constitutes an MPE.

<u>Case II</u>: $\bar{q} < q^m$

By the same argument as in case I, $R(q) \geq \bar{q}$ for all $q < \bar{q}$. In particular, since monopoly profit is the highest conceivable profit per period, $R(0) = q^m$.

Now suppose that for $q < \bar{q}$, r is a realization of $R(q)$ but $r \neq \max\{T(q), \bar{q}\}$, where $T(q)$ is given by (7). Then $r \neq T(q)$, because $r \geq \bar{q}$. But since

$$V(q) = \pi(r,q) + \delta\pi(r,0) + \frac{\delta^2}{1-\delta}\pi^m,$$

it is clear that discounted profit could be raised by choosing $r'(\geq \bar{q})$ equal to r or $T(q)$. We conclude that

$$R(q) = \text{Max}\ \{\bar{q}, T(q)\}$$

for $q < \bar{q}$.

<u>Subcase A</u>: $\bar{q} \geq T(\bar{q})$

Then, for q less than $\bar{q}$

$$(15)\quad \pi(\bar{q},q) + \delta\pi(\bar{q},0) + \frac{\delta^2}{1-\delta}\pi^m > 0$$

The inequality reverses for $q > \bar{q}$. Hence $\bar{q} = q^{**}$, where q^{**} is the larger root of (8). From (8)

$$(16)\quad q^{**} = \frac{(1+\delta)d + \sqrt{(1+\delta)^2 d^2 + \frac{4(2+\delta)}{1-\delta}\left(f - \frac{\delta^2 d^2}{4}\right)}}{2(2+\delta)}$$

Because $\bar{q} < q^m$, we know from case I that $\delta < \delta_1$. But from (16) we know that there exists $\delta_2 \in (0,\delta_1)$ such that $q^{**} \geq T(q^{**}) = \frac{d}{2} - \frac{\bar{q}}{2(1+\delta)}$ holds if and only if $\delta \in [\delta_2,\delta_1)$. Thus, $\bar{q} > q^m$ and $\bar{q} \geq T(\bar{q})$ imply that $\delta \in [\delta_2,\delta_1)$, that $\bar{q} = q^{**}$, and that $R(q) = \max\{q^{**}, T(q)\}$. Now for $\delta \in (\delta_2,\delta_1)$, $q^{**} > T(q^{**})$, and there exists $\underline{q} < q^{**}$ such that $q > T(q)$ if and only if $q > \underline{q}$. Hence $R(q)$ takes the form (11). Furthermore, if $\delta \in [\delta_2,\delta_1)$ and $R(q)$ is defined by (11) it is straightforward to verify that R constitutes a MPE.

<u>Subcase B</u>: $\bar{q} < T(\bar{q})$

Then for $q < \bar{q}$

$$\pi(T(q),q) + \delta\pi(T(q),0) + \frac{\delta^2}{1-\delta}\pi^m > 0,$$

with the inequality reversed for $q < \bar{q}$. Hence, $\bar{q} = q^{***}$, the larger root of (9). By elimination, we conclude that if $\bar{q} < T(\bar{q})$ then $R(q)$ is defined by (12) and $\delta < \delta_2$. Conversely, one can easily check that for $\delta < \delta_2$, R defined by (12) constitutes a symmetric MPE.

Q.E.D.

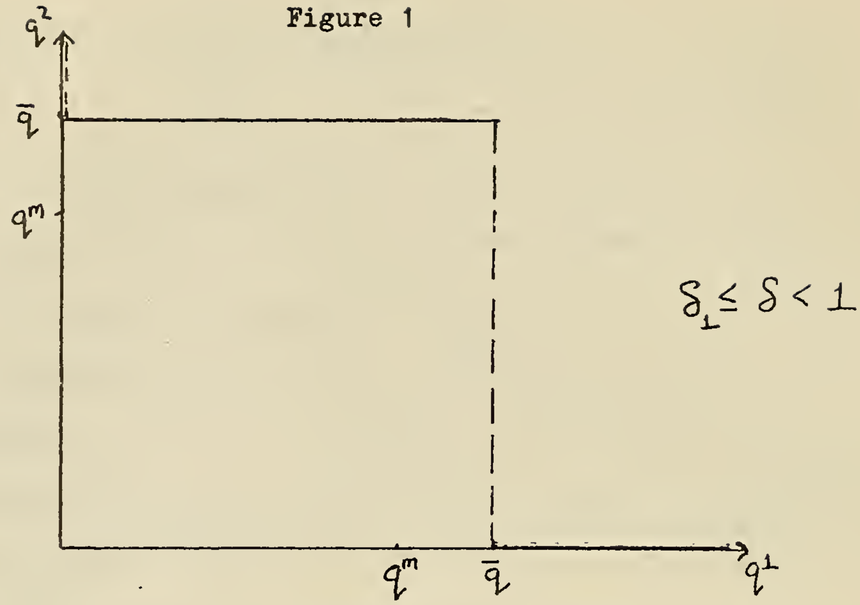
Proposition 2 shows that, regardless of the discount factor, equilibrium takes a simple form. Namely, there is a deterrence level $\bar{q}$, such that if a firm's rival is currently producing at or above this level, the firm will produce nothing. However, if the rival falls short of $\bar{q}$, the firm will produce at least at the level $\bar{q}$. Thus, there is a unique steady-state outcome wherein the single firm in the market produces at the level

$\max\{\bar{q}, q^m\}$. Moreover, starting from any other position, that steady-state is reached in a maximum of three periods.

The deterrence level $\bar{q}$ monotonically increases in the discount factor δ (and decreases in the fixed cost f). When δ is comparatively high (greater than δ_1), $\bar{q}$ is above the monopoly quantity q^m(see Figure 1). That is, to drive out its rival or deter it from entering, a firm must produce more than the monopoly quantity and, therefore, charge less than monopoly price. Given these restrictions, the firm will produce exactly $\bar{q}$. This is a result strongly reminiscent of the limit pricing literature (see Gaskins (1971), Kamien and Schwartz (1971), and Pyatt (1971)): an incumbent firm sells at a price sufficiently low that the immediate short-run losses of entry outweigh the longer run gains. Since $\bar{q}$ must satisfy (6) notice that, as δ tends to 1, $\pi^1(\bar{q},0)$ tends to zero. That is, instantaneous profit is driven down to the competitive level. Hence our model confirms the heuristic stories of Grossman (1981) and Baumol, Panzar, and Willig (1982) if firms place sufficient weight on future profits.

When the discount factor is less than δ_1, the deterrence level is below the monopoly level (see Figures 2 and 3). Hence, the steady quantity is the monopoly level itself, a result in keeping with the barriers to entry tradition. <u>How</u> a firm takes over the market from its rival depends on the discount factor and the rival's quantity, q. The firm could always drive out the rival by playing $\bar{q}$. However, for moderate discount factors ($\delta_2 < \delta < \delta_1$) and low values of q or for low discount factors ($\delta < \delta_1$) and <u>any</u> q (less than $\bar{q}$), the firm prefers to produce <u>more</u> than $\bar{q}$, namely T(q). T(q), defined by (7), can be thought of as the optimal "two-period reaction" function. It is a firm's best response to q in a game with a two-period horizon, given that the other firm does not produce in the second period.

Figure 1



Dotted lines denote firm 1's reaction function;
Solid lines denote firm 2's reaction function.

Figure 2

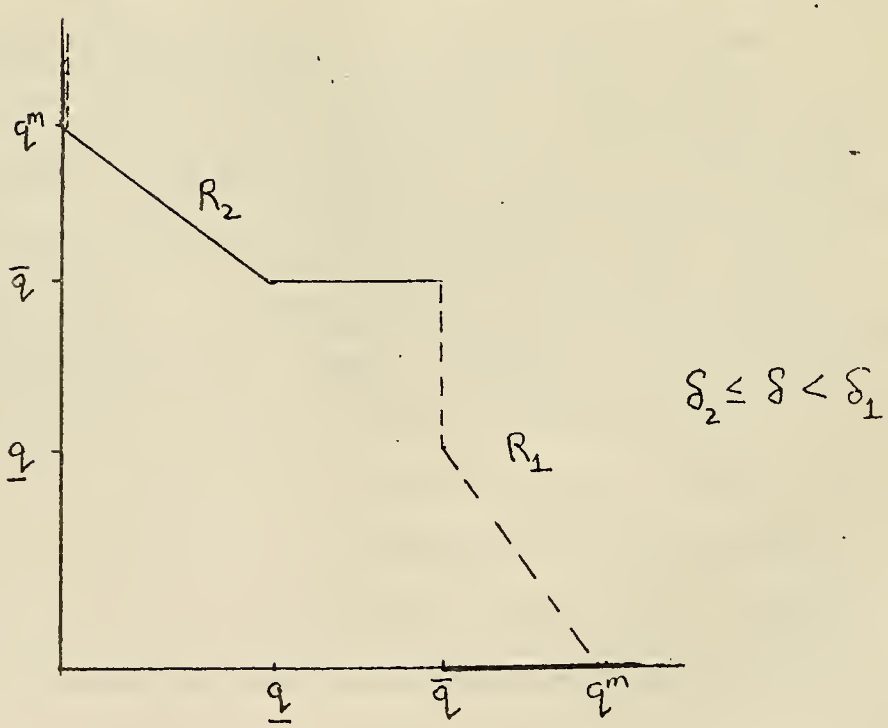
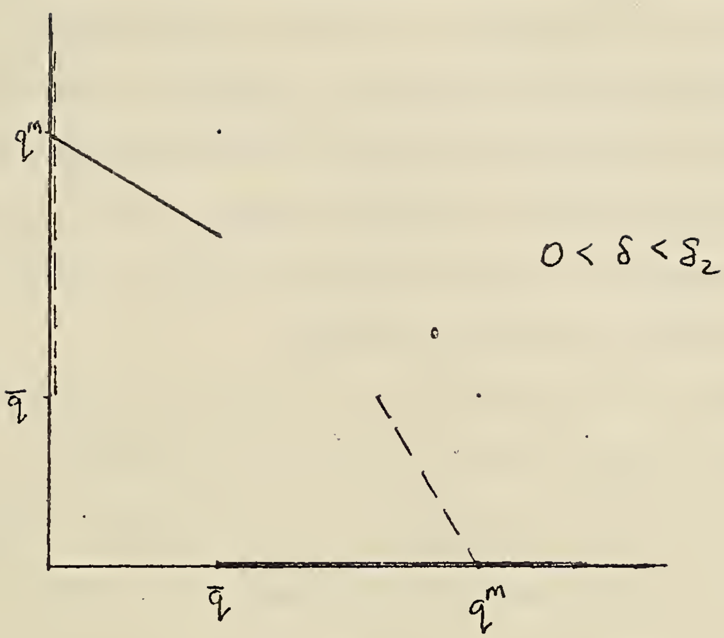


Figure 3



We should emphasize the role that perfection plays in Proposition 2. If we required only that reaction functions be in Nash equilibirum for a given starting point, we would find many symmetric equilibria besides the one we exhibited. For example, regardless of the discount factor, a firm could play the monopoly output (assuming the other firm were out of the market) in equilibrium by threatening to flood the market if the other firm ever dared enter. Since this bluff would never be called, it could be part of each firm's equilibrium strategy.

We ought also mention that although Proposition 3 exhibits the unique symmetric MPE, there are also, for sufficiently large fixed costs and discount factors, two highly asymmetric MPE's. Specifically, for such costs and discount factors, if firm 1 always uses its two-period reaction function, then firm 2 will always stay out of the market. Conversely, if firm 2 never enters, the two-period reaction function is optimal for firm 1. Thus this pair of strategies is a MPE, and so is the pair with the roles of the players interchanged. Moreover, these are the only two asymmetric equilibria.

Proposition 3: There exist $\underline{\delta} \in (0,1)$ and $\underline{f} \leq \pi^m$ such that if $\underline{\delta} \leq \delta < 1$ and $\underline{f} \leq f < \pi^m$, there are exactly two asymmetric equilibria:

$$(R^1, R^2) = (T, 0)$$

and

$$(R^1, R^2) = (0, T),$$

where T satisfies (7).

Proof: See the Appendix.

7. Quantity Competition with Large Fixed Costs: Endogenous Timing

In this section we examine exactly the same questions as in Section 6, but with our endogenous timing model (see Section 3). Let marginal costs, fixed costs, and instantaneous profit functions be exactly as in the last section (formulas 1 and 2). Moves, as before, consist of choosing nonnegative values of q. If a firm chooses $q > 0$ it cannot move again for two periods. If it chooses $q = 0$, however, it can move in the following period but must incur an entry cost e if, at that point, it takes $q > 0$.

Again we are interested in the Markov perfect equilibria. Now, however, the payoff relevant state is more complex; for the firm about to move, it matters whether the other firm moved the previous period; if so, how; and which of the two firms is currently in the market. Hence, Markov strategies themselves are more complicated.

We shall call a firm "in the market" in a given period if it chose q to be positive in either of the two previous periods. Otherwise, the firm is "out" (meaning that it took $q = 0$ the previous period). To simplify matters, let

$$A = [0,\infty)$$

$$A^* = [0,\infty)\ \{i\},$$

and let a firm's reaction function be described by a pair of (possibly random) functions

$$R_i: \ A^* \to A$$

$$R_0: \ A^* \to A$$

(We shall delete all superscripts since we shall work exclusively with symmetric equilibrium). The function "R_i" pertains to a firm that is currently "in" the market (i.e., one that two periods previously chose a positive quantity). For $q \geq 0$, $R_i(q)$ is just the reaction by a firm currently in the market

if the other firm (also) moved two periods ago. Similarly $R_0(q)$ and $R_0(i)$ are reactions by a firm currently out of the market.

The following result is our principal observation.

Proposition 4: There exists $\delta_* < 1$ such that for all $\delta > \delta_*$ and $\frac{1}{2}\pi^m < f < \pi^m$, the following is a symmetric Markov perfect equilibrium of the endogenous timing version of the model of Section 6. Furthermore, for $e > \pi^m(1+\delta)$ and $\frac{2}{3}\pi^m < f$, it is the unique symmetric Markov perfect equilibrium.

$$R_i(q) = \begin{cases} \bar{q}, \text{ if } q = 0 \\ \left.\begin{cases} \bar{q} \text{ with probability } \alpha_i \\ \\ 0 \text{ with probability } 1-\alpha_i, \end{cases}\right\} \text{ if } q = i \\ \bar{\bar{q}}, \text{ if } 0 < q < \bar{\bar{q}} \\ 0, \text{ if } q \geq \bar{\bar{q}} \end{cases}$$

$$R_0(q) = \begin{cases} \left.\begin{cases} \bar{q} \text{ with probability } \alpha_0 \\ \\ 0 \text{ with probability } 1-\alpha_0, \end{cases}\right\} \text{ if } q = 0 \\ 0, \text{ if } q = i \\ \bar{\bar{q}}, \text{ if } 0 < q < \bar{q} \\ 0, \text{ if } q \geq \bar{q}, \end{cases}$$

where $\bar{\bar{q}} > \bar{q} > q^m$ and, given $q_1 = \bar{q}$, $q_2 = \bar{\bar{q}}$ is the larger root of

$$(17)\quad \pi(q_2, q_1) - e + \delta\pi(q_2, 0) + \frac{\delta^2}{1-\delta}\pi(q_1, 0) = 0$$

and

$$(18)\quad \pi(q_2, q_2) + \delta\pi(q_2, 0) + \frac{\delta^2}{1-\delta}\pi(q_1, 0) = 0,$$

and α_0 and α_i satisfy $0 < \alpha_0, \alpha_i < 1$,

(19) $(1-\alpha_0)(\frac{\pi(\bar{q},0)}{1-\delta} - e) + \alpha_0(\pi(\bar{q},\bar{q})(1+\delta) - e) = 0$

and

(20) $(1-\alpha_i)(\frac{\pi(\bar{q},0)}{1-\delta}) + \alpha_i(\pi(\bar{q},\bar{q})(1+\delta)) = 0.$

Proof: See the appendix.

This equilibrium is evidently more complex than that of Proposition 2. However, it shares much in common with that earlier equilibrium (more specifically, with (10), the equilibrium that obtains for high discount factors). First, and most important, the nature of the steady state is the same: one firm stays out forever, and the other produces at a deterrence level $\bar{q}$ that falls with f and tends to the zero-profit level as δ tends to 1. Furthermore, the behavior of a firm when its rival has just played a positive q is very similar. Depending on whether the firm is currently in or out of the market, it plays deterrence level $\bar{\bar{q}}$ for all q less than $\bar{\bar{q}}$ or $\bar{q}$, respectively. For larger q it stays out altogether. Finally, although convergence to the steady state need not take place in only three steps - in fact, because the equilibrium is stochastic there is no guaranteed upper bound - it is nonetheless guaranteed in finite time with probability one.

The hypothesis $e > \pi^m(1+\delta)$ is important for the conclusion of Proposition 4. Without it, other symmetric equilibria are possible. We have constructed two. In one, the steady state consists of the two firms taking turns playing the monopoly level q^m. In the other, there is no deterministic steady-state at all. We conjecture that these two equilibria together with the one of Proposition 4, are the only symmetric equilibria.

8. CONCLUDING REMARKS

This paper and its two sequels represent an approach to the dynamics of oligopoly that differs philosophically rather strongly from the only full-fledged dynamic theory already in existence, the theory of supergames. Our theory, however, is not nearly so well-developed as the supergame literature. Much work remains to be done.

We feel that two modifications of our model are particularly important. One is to adopt continuous time. An important limitation of the current model is that the only possible relative timings are simultaneity and equal-spaced alternation. In particular, we do not allow the possibility of a firm's moving just momentarily after its rival. Such behavior could be accommodated in continuous time (or, approximately, in discrete-time with a period of commitment longer than two periods). How, if at all, such a modification would change the nature of equilibrium remains an open question.

Of course, it seems likely that, even in continuous time, equilibrium will continue to look the same if we postulate a reaction lag on the part of the rival of exactly one-half the commitment period. Moving any sooner would then amount to moving simultaneously. But before resorting to such assumptions, however plausible they may be, we must understand the properties of equilibrium when they are not made.

The other important potential modification is to generalize the costs of adjustment. Our current costs are extreme. The cost of changing ones' move before the commitment period has elapsed is, in effect, infinite; it then drops to zero. The effect of introducing a smoother schedule seems well worth investigating.

APPENDIX

Proposition 3: There exist $\underline{\delta} \varepsilon (0,1)$ and $\underline{f} < \pi^m$ such that if $\underline{\delta} \leq \delta < 1$ and $\underline{f} \leq f < \pi^m$, there are exactly two asymmetric equilibria:

$$(R^1, R^2) = (T, 0)$$

and

$$(R^1, R^2) = (0, T),$$

where T satisfies (7)

Proof: We begin by observing that, for δ sufficiently high and $f \geq \frac{5}{7}\pi^m$, firm 1 must lose money over the two periods if it plays positively the first period and firm 2 responds with its "two-period reaction function." That is,

$$\text{(A1)} \quad \sup_{q>0} \{(\pi^1(q,0)-f) + \delta(\pi^1(q,T(q))-f)\} < 0.$$

Substituting for π^1 in the left-hand side of (A1) and using $T(q) = \frac{d}{2} - \frac{q}{2(1+\delta)}$, we obtain $\max\{q(d-q)-f+\delta(q(\frac{d}{2}-q+\frac{q}{2(1+\delta)})-f)\}$, which for $\delta=1$ is negative if

$$\text{(A2)} \quad f > \frac{9}{56} d^2 = \frac{9}{14} \pi^m$$

Hence, by continuity, (A1) holds for δ in a neighborhood of 1 if $f > \frac{5}{7}\pi^m$.

We next establish, for sufficiently high δ and f, the counterpart of Lemma 5 for asymmetric equilibria.

Claim: For sufficiently high $\delta < 1$ and $f < \pi^m$, $R^2(r) = 0$ for all positive realizations r of $R^1(q)$ and all q, if R^1 and R^2 are equilibrium dynamic reaction functions.

Proof of Claim: Suppose that, to the contrary, for some q there existed a positive realization r_1 of $R^1(q)$ and positive realization r_2 of $R^2(r_1)$. From Lemma 2 (which applies to all, not just symmetric, MPE's), $R^2(r_1) > 0$. Thus we may suppose that r_2 is the lowest possible realization of $R^2(r_1)$. (If $R^2(r_1)$ does not literally have a lowest realization, then suppose that r_2 is "sufficiently" near the infimum of $R^2(r_1)$). Continuing iteratively, let r_3 be the infimum of $R^1(r_2)$, r_4 the infimum of $R^1(r_3)$, etc. Suppose first that all the r's are positive.

From Lemma 1, the subsequence $\{r_1, r_3, r_5, \ldots\}$ is nondecreasing (nonincreasing), while the subsequence $\{r_2, r_4, r_6, \ldots\}$ is nonincreasing (nondecreasing). In either case there exists $\bar{n}$ such that either, for all n and n' greater than $\bar{n}$, $r_{2n} \leq r_{2n'+1}$ or else, for all n and n' greater than $\bar{n}$, $r_{2n+1} \leq r_{2n'}$. In either case, for one of the two firms, the present discounted value of profits starting in period $\bar{n}$ is necessarily negative, since the infimum of the other firm's quantity is always at least as great as its own. Thus, eventually one of the r_n's $(n > 2)$ must be zero. Suppose, for simplicity, that it is r_3. If $r_2 < T(r_1)$, then firm 2 could raise its profit by substituting $T(r_1)$ for r_2 (since it would a fortiori induce firm 1 to leave the market). Hence r_2 must be greater than equal $T(r_1)$. But then, from (A1), firm 1 must lose money by playing r_1. Hence the claim is established.

Consider an asymmetric Markov perfect equilibrium. For each i, let $\bar{q}^i$ $= \inf\{q \mid R^j(q) = 0\}$ (conceivably $\bar{q}_i$ could equal 0 or ∞), $j \neq i$. Suppose first that $0 < \bar{q}^1, \bar{q}^2 < \infty$. Let $\tilde{q}^1 = \lim\sup_{\varepsilon \to 0} \{r \mid r \varepsilon R^1(\bar{q}^2 - \varepsilon)\}$. By definition of $\bar{q}^2$, $r^1(\varepsilon) \equiv \sup\{r \mid r \varepsilon R^1(\bar{q}^2 - \varepsilon)\} > 0$ for any $\varepsilon > 0$. Hence from the above

claim and Lemma 1, $R^2(r^1(\varepsilon)) = 0$, and so $\tilde{q}^1(\varepsilon) \geq \overline{q}^1$. We conclude that $\tilde{q}^1 \geq \overline{q}^1$. If $\tilde{q}^1 > q^m$, then $\tilde{q}^1 = \overline{q}^1$, otherwise for small ε, firm 1 could reduce its output, raise its short-period profit and still deter entry in response to $\overline{q}^2 - \varepsilon$. If $\tilde{q}^1 \leq q^m$, then $\overline{q}^1 \leq q^m$, and so $\tilde{q}^1 = \max\{\overline{q}^1, T(\overline{q}^2)\}$.

There are therefore three possibilities:

(i) $\tilde{q}^1 = \overline{q}^1 > q^m$

(ii) $q^m \geq \tilde{q}^1 = \overline{q}^1 \geq T(\overline{q}^2)$

or

(iii) $q^m \geq \tilde{q}^1 = T(\overline{q}^2) > \overline{q}^2$.

We first rule out cases (ii) and (iii). If, to the contrary, either case holds, then if firm 2 plays $q^m + \varepsilon$, firm 1 will not produce. Hence firm 2 can earn

essentially monopoly profit indefinitely. Thus for high discount factors $\overline{q}^2 > q^m$. If $\overline{q}^2 = \infty$, Then

$$\lim_{q \to \infty} \left[(d - q - \tilde{q}^1)\tilde{q}^1 + \delta(d - \tilde{q}^1)\tilde{q}^1 + \frac{\delta^2}{1-\delta}\pi^m - \frac{f}{1-\delta} \right] \geq 0$$

which is plainly false. Hence $\overline{q}^2 < \infty$. Then

(A3) $(d-\overline{q}^2-\tilde{q}^1)\tilde{q}^1 + \delta(d-\tilde{q}^1)\tilde{q}^1 + \frac{\delta^2}{1-\delta}\pi^m - \frac{f}{1-\delta} = 0.$

Similarly,

(A4) $(d-\overline{q}^1-\overline{q}^2)\overline{q}^2 + \frac{\delta}{1-\delta}(d-\overline{q}^2)\overline{q}^2 - \frac{f}{1-\delta} = 0.$

Now as δ tends to 1, $\overline{q}^2$ must grow indefinitely if (A3) is to hold. But (A4) clearly cannot hold for arbitrarily large $\overline{q}^2$, and so, for large δ, cases (ii) and (iii) are impossible.

We conclude that either

(iv) $\overline{q}^i > q^m$, $i = 1, 2$

or

(v) $\overline{q}^1 = 0$ or $\overline{q}^2 = 0$.

If (iv) holds then

(A5) $$(d-\overline{q}^1-\overline{q}^2)\overline{q}^1 + \frac{\delta}{1-\delta}(d-\overline{q}^1)\overline{q}^1 - \frac{f}{1-\delta}$$
$$= (d-\overline{q}^1-\overline{q}^2)\overline{q}^2 + \frac{\delta}{1-\delta}(d-\overline{q}^2)\overline{q}^2 - \frac{f}{1-\delta}$$
$$= 0,$$

which implies $\overline{q}^1 = \overline{q}^2$, violating asymmetry.

Thus (v) must hold. If $\overline{q}^2 = 0$, then $R^2(q) = T(q)$. From (A1), we deduce that $R^1(q) = 0$. Similarly $\overline{q}^1 = 0$ implies that $(R^1, R^2) = (T, 0)$.

Q.E.D.

Proposition 4: There exists $\underline{\delta} < 1$ such that for all $\delta > \underline{\delta}$, $e > \pi^m(1+\delta)$ and $\frac{2}{3}\pi^m < f < \pi^m$, the following is the unique symmetric MPE

$$R_i(q) = \begin{cases} \bar{q}, & q = 0 \\ \begin{cases} \bar{q}, \text{ with probability } \alpha_i \\ 0, \text{ with probability } 1-\alpha_i \end{cases} & q = i \\ \bar{\bar{q}}, & 0 < q < \bar{\bar{q}} \\ 0, & q \geq \bar{\bar{q}} \end{cases}$$

$$R_0(q) = \begin{cases} \begin{cases} \bar{q} \text{ with probability } \alpha_0 \\ 0 \text{ with probability } 1-\alpha_0 \end{cases} & q = 0 \\ 0, & q = i \\ \bar{\bar{q}}, & 0 < q < \bar{q} \\ 0, & q \geq \bar{q} \end{cases}$$

where $\bar{q} > q^m$ and, given $q_1 = \bar{q}$, $q_2 = \bar{\bar{q}}$ is the larger root of

(17) $\pi_0(q_2, q_1) + \delta\pi_i(q_2, 0) + \frac{\delta^2}{1-\delta}\pi_i(q_1, 0) = 0$

and

(18) $\pi_i(q_2, q_2) + \delta\pi_i(q_2, 0) + \frac{\delta^2}{1-\delta}\pi_i(q_1, 0) = 0$

and α_0 and α_i satisfy $0 < \alpha_0, \alpha_i < 1$,

(19) $(1-\alpha_0)(\frac{\pi_i(\bar{q},0)}{1-\delta} - e) + \alpha_0(\pi_i(\bar{q},\bar{q})(1+\delta)-e) = 0$

and

(20) $(1-\alpha_i)(\frac{\pi_i(\bar{q},0)}{1-\delta}) + \alpha_i(\pi_i(\bar{q},\bar{q})(1+\delta)) = 0.$

The proof begins with a series of Lemmas. To establish uniqueness, we assume the existence of a symmetric equilibrium (R_i,R_0). We proceed to show that (R_i,R_0) must be defined as in the statement of the proposition.

Lemma A1: R_i and R_0 are nonincreasing for $q > 0$.

The proof is the same as that of Lemma 1.

Lemma A2: If 0 if a realization of $R_i(q)$ and $q > 0$ is a realization of either $R_i(q')$ or $R_0(q')$ for some q', then $R_i(q) = 0$.

The proof is the same as that of Lemma 2.

Lemma A3: If r is a positive realization of $R_0(q)$ and $q > 0$ then $\pi_i(r,q)+\delta W_i(r) \geq \pi_i(r',q)+\delta W_i(r')$ for all r' where $W_i(x)$ is the value of having just played x when the other firm is currently in the market.

Proof: The hypothesis imply that for all r'

(i) $\pi_0(r,q)+\delta W_i(r) \geq \pi_0(r',q)+\delta W_i(r')$.

From (i) we deduce:

(ii) $\pi_0(r,q) - \pi_0(r',q) \geq \delta(W_i(r')-W_i(r))$. For $r' > 0$,

(iii) $\pi_i(r,q) - \pi_i(r',q) = \pi_0(r,q)-\pi_0(r',q)$

and, for $r' = 0$,

(iv) $\pi_i(r,q)-\pi_i(0,q) > \pi_0(r,q)-\pi_0(0,q)$.

Combining (ii)-(iv) and rearranging we obtain

(v) $\pi_i(r,q) + \delta W_i(r) \geq \pi_i(r',q)+\delta W_0(r')$

for all r'.

Q.E.D.

Lemma A4: If for $q > 0$ r is a positive realization of $R_i(q)$ or $R_0(q)$, then $r > q$.

The proof is the same as in the fixed timing case if r is a realization of $R_i(q)$. If r is a realization of $R_0(q)$, then from Lemma A3, r is a realization of $R_i(q)$ and then the argument is the same as before.

Q.E.D.

Lemma A5: There exist nonnegative $\bar{q}$ and $\bar{\bar{q}}$ such that, for all $q > \bar{\bar{q}}$, $R_i(q) = 0$; for all $q > \bar{q}$, $R_0(q) = 0$; for all $q \in (0,\bar{\bar{q}})$, $R_i(q)$ has positive realizations; and for all $q \in (0,\bar{q})$, $R_0(q)$ has positive realizations.

The proof is the same as that of Lemma 4.

Lemma A6: For all $q > 0$ and all positive realizations r of either $R_i(q)$ or $R_0(q)$, $R_i(r) = 0$.

The proof is the same as that of Lemma 5.

Lemma A7: For all $q \in (0,\bar{\bar{q}})$, $R_i(q) = \max\{T(q),\bar{\bar{q}}\}$ and for all $q \in (0,\bar{q})$ $R_0(q) = \max\{T(q),\bar{\bar{q}}\}$.

From a given firm's standpoint, there are four possible states of the game if neither firm is currently committed to an action: (0,0), (0,i), (i,0), (i,i), (the first term refers to the firm under consideration and the second to the other firm; 0 = "out" and i = "in"). We next show that in any of these states, a firm randomizes among at most three different quantities.

Lemma A8: For either (0,0) or (i,0) (resp. (0,i) or (i,i)), there exist $\hat{q}$ with $\hat{q} \in (0,\bar{q})$ if $\bar{q} > 0$ (resp., $\hat{q} \in (0,\bar{\bar{q}})$ if $\bar{\bar{q}} > 0$) and $\hat{\hat{q}} \geq \bar{q}$ (resp., $\hat{\hat{q}} \geq \bar{\bar{q}}$) such that

$$\operatorname{supp} R(q) \subset \{0,\hat{q},\hat{\hat{q}}\}.$$

Proof: We prove the Lemma for state (0,0). The proofs for other states are similar. Let $\overline{p}$ be the probability that $R_0(0) > 0$. If there exists a positive realization q^+ of $R_0(0)$ such that $q^+ < \overline{q}$ then

$$\text{(A6)}\quad q^+ = \underset{q \in (0,\overline{q})}{\arg\max} \Big[(1-\overline{p})\pi_0(q,0) + (1-\overline{p})\delta\, E\pi_i(q,R_0(q)) + (1-\overline{p})\delta^3 V(0,i) + \overline{p}\, E_x(\pi_0(q,x)+\delta\pi_i(q,x)) + \overline{p}\,\delta^2 V(i,i)\Big]$$

where

$$\text{(A7)}\quad \pi_i(a,b) = (d-a-b)a - f$$

$$\text{(A8)}\quad \pi_0(a,b) = \pi_i(a,b) - e,$$

and, from Lemma A7,

$$\text{(A9)}\quad R_0(q) = \max\{\overline{q},\ T(q)\}$$

From (A6)-(A9), the bracketed expression on the right hand side of (A6) is the minimum of a sum of quadratics. Hence the maximizer of this expression is unique, and q^+ is well-defined. The argument is similar for $q^+ \geq \overline{q}$.

Q.E.D.

Let us adopt the notation of the following tables:

	(0,0)	(0,i)	(i,0)	(i,i)
$\hat{q}$	q^*	q_0^*	q_i^*	q_j^*
$\hat{\hat{q}}$	q^{**}	q_0^{**}	q_i^{**}	q_j^{**}

and

	(0,0)	(0,i)	(i,0)	(i,i)
Probability(R=q)	p^*	p_0^*	p_i^*	p_j^*
Probability(R=q)	p^{**}	p_0^{**}	p_i^{**}	p_j^{**}

<u>Lemma A9</u>: In a symmetric equilibrium, $p_0^* = p_j^* = 0$.

<u>Proof</u>: We begin by establishing some formulae for values. First

(A10) $$V(0,0) = \delta(1-p^*-p^{**})V(0,0) + p^*(\delta\pi_0(R_0(q^*),q^*) + \delta^2\pi_i(R_0(q^*),0) + \delta^3 V(i,0)) + p^{**}\delta^2 V(0,i), \text{ if } 1-p^*-p^{**} > 0$$

Because $e > \pi^m(1+\delta)$, $\pi_0(R_0(q^*),q^*) + \delta\pi_i(R_0(q^*),0) < 0$.

Therefore (A10) implies that if $1-p^*-p^{**} > 0$ either

(A10a) $V(0,0) = 0$ (if $p^* = p^{**} = 0$)

or

(A10b) $V(0,0) < \max\,(V(i,0),V(0,i))$.

Next

(A11) $$V(0,0) = (1-p^*-p^{**})(\pi_0(q^*,0) + \delta\pi_i(q^*,R_0(q^*)) + \delta^3 V(0,i)) + p^*(\pi_0(q^*,q^*) + \delta\pi_i(q^*,q^*) + \delta^2 V(i,i)) + p^{**}(\pi_0(q^*,q^{**}) + \delta\pi_i(q^*,q^{**}) + \delta^2 V(i,i)),$$

if $p^* > 0$

Because $e > \pi^m(1+\delta)$, $\pi_0(q^*,0) + \delta\pi_i(q^*,R_0(q^*))$,

$\pi_0(q^*,q^*) + \delta\pi_i(q^*,q^*)$, and $\pi_0(q^*,q^{**}) + \delta\pi_i(q^*,q^{**})$

are all negative. Hence (A11) implies

(A11a) $V(0,0) < \max\,(V(0,i),\ V(i,i))$ if $p^* > 0$

Also, if $p^{**} > 0$,

$$\text{(A12)}\quad V(0,0) = (1-p^*-p^{**})(\pi_0(q^{**},0) + \delta\pi_i(q^{**},0) + \delta^2 V(i,0)) + p^*(\pi_0(q^{**},q^*) + \delta\pi_i(q^{**},q^*) + \delta^2 V(i,i)) + p^{**}(\pi_0(q^{**},q^{**}) + \delta\pi_i(q^{**},q^{**}) + \delta^2 V(i,i))$$

From (A12) and the fact that $e > \pi^m(1+\delta)$ we have

(A12a) $V(0,0) < \max(V(i,0),V(i,i))$ if $p^{**} > 0$.

Also

$$\text{(A13)}\quad V(0,i) = (1-p_i^*-p_i^{**})\delta V(0,0) + p_i^*(\delta\pi_0(R_0(q_i^*),q_i^*) + \delta^2\pi_i(R_0(q_i^*),0) + \delta^3 V(i,0)) + p_i^{**}\delta^2 V(0,i),$$

if $1-p_0^*-p_0^{**} > 0$.

Thus, from (A13) and $e > \pi^m(1+\delta)$ we have either

(A13a) $V(0,i) \leq V(0,0)$, if $p_i^* = 0$

or

(A13b) $V(0,i) < \max(V(i,0),V(0,0))$, otherwise

if $1-p_0^*-p_0^{**} > 0$

Also, if $p_0^* > 0$

$$\text{(A14)}\quad V(0,i) = (1-p_i^*-p_i^{**})(\pi_0(q_0^*,0) + \delta\pi_i(q_0^*,R_0(q_0^*)) + \delta^3 V(0,i)) + p_i^*(\pi_0(q_0^*,q_i^*) + \delta\pi_i(q_0^*,q_i^*) + \delta^2 V(i,i)) + p_i^{**}(\pi_0(q_i^*,q_i^{**}) + \delta\pi_i(q_0^*,q_i^{**}) + \delta^2 V(i,i)),$$

Therefore

(A14a) $V(0,i) < V(i,i)$, if $p_0^* > 0$

Also, if $p_0^{**} > 0$,

$$\text{(A15)}\quad V(0,i) = (1-p_i^*-p_i^{**})(\pi_0(q_0^{**},0) + \delta\pi_i(q_0^{**},0) + \delta^2 V(i,0)) + p_i^*(\pi_0(q_0^{**},q_i^*) + \delta\pi_i(q_0^{**},q_i^*) + \delta^2 V(i,i)) + p_i^{**}(\pi_0(q_0^{**},q_i^{**}) + \delta\pi_i(q_0^{**},q_i^{**}) + \delta^2 V(i,i)),$$

if $p_0^{**} > 0$

Therefore,

(A15a) $V(0,i) < \max(V(i,0), V(i,i))$, if $p_0^{**} > 0$.

Also

(A16) $V(i,0) = (1-p_0^*-p_0^{**})\delta V(0,0) + p_0^*(\delta\pi_0(R_0(q_i^*),q_0^*)$
$+ \delta^2\pi_i(R_0(q_j^*),0) + \delta^3 V(i,0))$
$+ p_0^{**}\delta^2 V(0,i),$ if $1-p_i^*-p_i^{**} > 0.$

Therefore,

(A16a) $V(i,0) < \max(V(0,0),V(0,i)),$ if $1-p_i^*-p_i^{**} > 0$

Also

(A17) $V(i,i) = (1-p_j^*-p_j^{**})\delta V(0,0) + p_j^*(\delta\pi_0(R_0(q_j^*),q_j^*)$
$+ \delta^2\pi_i(R_0(q_j^*,0) + \delta^3 V(i,0)) + p_j^{**}\delta^2 V(0,i),$
if $1-p_j^*-p_j^{**} > 0.$

Now if $1-p_j^*-p_j^{**} = 0$, then, starting from (i,i), both firms stay in the market forever and, hence, make negative profits. Therefore, $1-p_j^*-p_j^{**} > 0$, and so (A17) always holds. We conclude that either

(A17a) $V(i,i) < \max(V(0,0),V(i,0),V(0,i)).$

or

(A17b) $V(i,i) = V(0,0) = V(0,i) = 0$ and $p_j^* = 0.$

Also

(A18) $V(i,i) = (1-p_j^*-p_j^{**})(\pi_i(q_j^*,0) + \delta\pi_i(q_j^*,R_0(q_j^*))$
$+ \delta^3 V(0,i)) + p_j^*(\pi_i(q_j^*,q_j^*)(1+\delta) + \delta^2 V(i,i))$
$+ p_j^{**}(\pi_i(q_j^*,q_j^{**})(1+\delta) + \delta^2 V(i,i)),$ if $p_j^* > 0.$

Now $\max_q[(d-q)q + (d-2q)q] = \frac{d^2}{3}.$

Therefore, if $f > \frac{d^2}{6} = \frac{2}{3}\pi^m$ and δ is sufficiently close to 1,

(A18a) $\pi_i(q_j^*,0) + \delta\pi_i(q_j^*,R_0(q_j^*)) < 0.$

Because $1-p_j^*-p_j^{**} > 0$, (A18) implies

(A18b) $V(i,i) < V(0,i),$ if $p_j^* > 0$

Finally,

(A19) $V(i,0) = (1-p_0^*-p_0^{**})(\pi_i(q_i^*,0) + \delta\pi_i(q_i^*,R_0(q_i^*)) + \delta^3 V(0,i))$
$+ p_0^*(\pi_i(q_i^*,q_0^*)(1+\delta) + \delta^2 V(i,i))$
$+ p_0^{**}(\pi_i(q_i^*,q_0^{**})(1+\delta) + \delta^2 V(i,i)),$ if $p_i^* > 0$

<u>Case I</u>: $p_0^* > 0$ and $p_j^* > 0$

But (A14a) and (A18b) contradict each other. So Case I is impossible.

<u>Case II</u>: $p_0^* > 0 \quad p_j^* = 0$

From (A17a) and (A17b)

(A20) $V(i,i) \leq \max\,(V(0,0), V(0,i))$.

Combining (A14a) and (A20) we obtain

(A21) $V(0,i) < V(i,i) \leq V(0,0)$

If $p^* > 0$, then (A11a) contradicts (A21). Therefore $p^* = 0$. Notice that (A21) implies that $V(0,0) > 0$. If $1-p^*-p^{**} > 0$, then (10b) and $p^* = 0$ imply that

$$V(0,0) < V(0,i),$$

contradicting (A21). Therefore $1-p^*-p^{**} = 0$. Then $p^{**} = 1$ (since $p^* = 0$). But (A12) then implies that $V(0,0) < V(i,i)$ contradicting (A21). Thus Case II is impossible.

<u>Case III</u>: $p_0^* = 0$, $p_j^* > 0$

Because $p_j^* > 0$, (A18b) holds.

Suppose first that $p_0^{**} > 0$. Then (A15a) holds. Combining (A15a) and (A18b), we have

(A22) $V(i,i) < V(0,i) < V(i,0)$.

If $1-p_i^*-p_i^{**} = 0$, then (A15) implies

$$V(0,i) < V(i,i),$$

contradicting (A22). Therefore, assume that $1-p_i^*-p_i^{**} > 0$. From (A16a) and (A22),

$$V(i,0) < V(0,0).$$

Thus

(A23) $V(i,i) < V(0,i) < V(i,0) < V(0,0)$

Now at least one of (A10b), (A11a), and (A12a) must hold. But all of them contradict (A23).

We conclude, therefore, that $p_0^{**} = 0$. Thus $1-p_0^*-p_0^{**} = 1$. If $p_i^{**} = 1$, then (A13) implies that

$$V(0,i) = \delta^2 V(0,i),$$

so that $V(0,i) = 0$. But this last conclusion contradicts (A18b). Therefore $p_i^{**} < 1$ and, since $V(0,i) > 0$, we have, from (A13a), (A13b), and (A18b)

(A24) $V(i,i) < V(0,i) \leq \max(V(i,0), V(0,0))$.

If $1-p_i^*-p_i^{**} > 0$, we have from (A16) and $1-p_0^*-p_0^{**} = 1$,

(A25) $V(i,0) < V(0,0)$.

Therefore from (A24) and (A25), $V(i,i) < V(0,i) \leq V(0,0)$ and $V(i,0) < V(0,0)$, which are contradicted by one of (A10b), (A11a), and (A12a). Thus assume $1-p_i^*-p_i^{**} = 0$. Then because $p_i^* > 0$ and $1-p_i^*-p_i^{**} = 0$, (A13) implies

(A26) $V(0,i) < V(i,0)$

From (A24) and (A26)

(A27) $V(i,i) < V(0,i) < V(i,0)$

Now $q_i^* < q_0^{**}$, and, therefore, $\pi_i(q_i^*,q_0^{**}) < 0$. Furthermore, as we already observed,

$$\pi_i(q_i^*,0) + \delta\pi_i(q_i^*,R_0(q_i^*)) < 0$$

Hence (A19) and $p_0^* = 0$ implies

$$V(i,0) < \max(V(0,i),V(i,i)),$$

which contradicts (A27). We conclude that Case III is impossible.

Therefore $p_0^* = p_j^* = 0$.

Q.E.D.

<u>Lemma A10</u>: In a symmetric equilibrium, $p^* = 0$, $p_i^{**} = 1$, and $1-p_0^*-p_0^{**} = 1$.

<u>Proof</u>: From (A17) and because, from Lemma A9, $p_j^* = 0$,

(A28) $V(i,i) < \max(V(0,0),V(0,i))$.

To show that $p^* = 0$, assume to the contrary that $p^* > 0$. Then (A11a) holds.

If $V(0,i) \leq V(i,i)$, then (A11a) implies that $V(0,0) < V(i,i)$, and so, from (A28), $V(i,i) < V(0,i)$, a contradiction. Therefore

(A29) $V(i,i) < V(0,i)$,

and from (A11a)

(A30) $V(0,0) < V(0,i)$.

Suppose first that $1-p_0^*-p_0^{**} > 0$. In view of (A30), (A13b) holds and implies

(A31) $V(0,i) < V(i,0)$.

Combining (A29)-(A31) we obtain

(A32) $\max(V(i,i),V(0,0),V(0,i)) < V(i,0)$.

If $1-p_i^*-p_i^{**} > 0$, then (A16a) contradicts (A32). Therefore assume $1-p_i^*-p_i^{**} = 0$. If $p_i^* > 0$ then (A19) implies

$$V(i,0) < \max(V(0,i),V(i,i)),$$

a contradiction of (A32). Therefore, assume $p_i^* = 0$. Then, since $1-p_i^*-p_i^{**} = 0$, $p_i^{**} = 1$. From (A13)

$$V(0,i) = \delta^2 V(0,i) = 0,$$

which contradicts (A29).

We conclude that $1-p_0^*-p_0^{**} = 0$. Thus $p_0^{**} = 1$. Hence (A15a) applies. From (A15a), (A29), and (A30),

(A33) $\max(V(0,0),V(0,i),V(i,i)) < V(i,0)$.

If $1-p_i^*-p_i^{**} > 0$, then (A16a) contradicts (A33). Therefore assume $1-p_i^*-p_i^{**} = 0$. If $p_i^* > 0$, then because $p_0^{**} = 1$ and $\pi_i(q_i^*,q_0^{**}) < 0$, (A19) implies $V(i,0) < V(i,i)$, a contradiction of (A33). Therefore assume $p_i^* = 0$, and thus, because $1-p_i^*-p_i^{**} = 0$, $p_i^{**} = 1$. Thus (A15) implies that $V(0,i) < V(i,i)$, a contradiction of (A29). We conclude that $p^* = 0$, after all.

To see that $p_i^{**} = 1$, assume, to the contrary, that $p_i^{**} < 1$.

Suppose first that $p_i^* > 0$. Then (A18a), $p_0^* = 0$, $\pi_i(q_i^*, q_0^{**}) < 0$, and (A19) imply

(A34) $V(i,0) < \max(V(0,i), V(i,i))$.

If $V(0,i) \leq V(i,i)$, then $V(i,i) > 0$, and from (A28) and (A34)

(A35) $V(i,0) < v(i,i) < V(0,0)$.

But if $p^{**} > 0$, then (A12a) contradicts (A35), and if $p^{**} = 0$ then (since $p^* = 0$) (A10a) contradicts (A35). Therefore assume that

(A36) $V(i,i) < V(0,i)$.

Then from (A34)

(A37) $V(i,0) < V(0,i)$.

If $V(0,i) \leq V(0,0)$, then (A36) and (A37) imply that $\max(V(i,i), V(i,0), V(0,i)) \leq V(0,0)$, which again contradicts either (A12a) or (A10b).

Therefore assume

(A38) $V(0,0) < V(0,i)$.

If $p_0^{**} < 1$, then (A13a) and (A13b) contradict (A37) and (A38).

Therefore assume $p_i^* = 0$. Since we are supposing that $p_i^{**} < 1$, (A16a) holds. If $V(0,0) < V(0,i)$, then (A16a) and (A28) imply

(A39) $\max(V(i,0), V(0,0), V(i,i)) < V(0,i)$.

But then (A13b) or (A15a) contradicts (A39), depending on whether $p_0^{**} < 1$ or $p_0^{**} = 1$. Thus assume

(A40) $V(0,i) \leq V(0,0)$.

From (A16a), (A28), and (A40),

(A41) $V(i,i) < V(0,0)$

From (A16a) and (A40)

(A42) $V(i,0) < V(0,0)$.

But either (A10b) contradicts (A40), or (A12a) contradicts (A42). We conclude that $p_i^{**} = 1$, afterall.

Finally, to establish that $p_0^{**} = 0$, suppose that $p_0^{**} > 0$. Then since $p_i^{**} = 1$, (A15) implies

(A43) $\quad V(0,i) < V(i,i)$.

From (A43) and (A28)

(A44) $\quad V(0,i) < V(i,i) < V(0,0)$.

Since $p^* = 0$, then from (A10)

$$V(0,0) < V(0,i)$$

contradicting (A44). Hence we conclude that $p_0^{**} = 0$ after all.

Q.E.D.

Using Lemmas A9 and A10 we can straightforwardly demonstrate that the equilibrium in the statement of Proposition 4 is the only symmetric equilibrium.

Proof of Proposition 4: We first show that in a symmetric equilibrium, for δ sufficiently close to 1, $\bar{\bar{q}} > q^m$. Suppose instead that $\bar{\bar{q}} \leq q^m$. For $\varepsilon > 0$, $R_i(\bar{\bar{q}}+\varepsilon) = 0$, and so, because $p_0^* = p_0^{**} = 0$ and $p_i^{**} = 1$, a firm (in the market) responding to $\bar{\bar{q}} + \varepsilon$ earns zero profit. If, however, the firm responds to $\bar{\bar{q}} + \varepsilon$ by playing $q^m + \gamma$ for $\gamma > 0$ the other firm will drop out of the market. Furthermore, by continuing to play $q^m + \gamma$ the firm earns discounted profit of

$$\pi_i(q^m+\gamma, \bar{\bar{q}}+\varepsilon) + \frac{\delta}{1-\delta}\pi_i(q^m+\gamma, \bar{\bar{q}}),$$

which, for δ sufficiently close to 1 and γ and ε small, is positive, a contradiction. Therefore $\bar{\bar{q}} > q^m$. From Lemma A7, we conclude that $R_i(q) = \bar{\bar{q}}$

for $q \varepsilon (0,\bar{q})$. If r is a positive realization of $R_i(\bar{q})$, then from the argument establishing Lemma A7, $r = \bar{q}$. But this contradicts Lemma A4. Hence $R_i(\bar{q}) = 0$. We conclude that

$$R_i(q) = \begin{cases} \bar{q}, & 0 < q < \bar{q} \\ 0, & q \geq \bar{q} \end{cases}$$

We next show that $\bar{q} > q^m$ for δ sufficiently close to 1. If $\bar{q} \leq q^m$, consider $R_0(\bar{q}+\varepsilon)$ for small $\varepsilon > 0$. By definition of $\bar{q}$, $R_0(\bar{q}+\varepsilon) = 0$. From Lemmas A9 and A10, therefore, the firm responding to $\bar{q} + \varepsilon$ earns a discounted sum of profits equal to zero (since $p_0^* = p_0^{**} = 0$ and $p_i^{**} = 1$). If, instead, the firm plays $\bar{q}$ its payoff is

(A50) $$\pi_0(\bar{q},\bar{q}+\varepsilon) + \delta\pi_i(\bar{q},0) + \frac{\delta}{1-\delta}\pi^m.$$

Hence, (A50) is nonpositive, and so

(A51) $$\pi_0(\bar{q},\bar{q}) + \delta\pi_i(\bar{q},0) + \frac{\delta^2}{1-\delta}\pi^m \leq 0.$$

Now $R_i(\bar{q}-\varepsilon) = \bar{q}$, and so $\pi_i(\bar{q},\bar{q}-\varepsilon) + \delta\pi_i(\bar{q},0) + \frac{\delta^2}{1-\delta}\pi^m$ is nonnegative for all $\varepsilon > 0$. Therefore

(A52) $$\pi_0(\bar{q},\bar{q}) + \delta\pi_i(\bar{q},0) + \frac{\delta^2}{1-\delta}\pi^m \geq 0.$$

Using (A51) and (A52), we have

$$\text{(A53)}\quad \pi_0(\bar{\bar{q}},\bar{q}) + \delta\pi_i(\bar{\bar{q}},0) + \frac{\delta}{1+\delta}\pi^m \le \pi_i(\bar{\bar{q}},\bar{\bar{q}}) + \delta\pi_i(\bar{\bar{q}},0) + \frac{\delta^2}{1+\delta}\pi^m$$

Simplifying we obtain

$$\bar{\bar{q}}^2 - \bar{q}\,\bar{\bar{q}} - e \le 0$$

Therefore

$$\text{(A54)}\quad \bar{\bar{q}} \le \frac{\bar{q} + \sqrt{\bar{q}^2 + 4e}}{2}.$$

But from (A54) and $\bar{q} \le q^m$, (A50) is positive for δ sufficiently close to 1, a contradiction of our earlier conclusion. Thus $\bar{q} > q^m$ after all.

By argument similar to that for R_i, we have

$$R_0(q) = \begin{cases} \bar{\bar{q}}, & 0 < q < \bar{q} \\ 0, & q \ge \bar{q}. \end{cases}$$

Therefore,

$$\text{(A35)}\quad \pi_i(\bar{\bar{q}},\bar{\bar{q}} + \varepsilon) + \delta\pi_i(\bar{\bar{q}},0) + \frac{\delta^2}{1-\delta}\pi_i(\bar{q},0) \begin{cases} \le 0, & \varepsilon \ge 0 \\ \ge 0, & \varepsilon \le 0 \end{cases}$$

and

$$\text{(A36)}\quad \pi_0(\bar{\bar{q}},\bar{q} + \varepsilon) + \delta\pi_i(\bar{\bar{q}},0) + \frac{\delta^2}{1-\delta}\pi_i(\bar{q},0) \begin{cases} \le 0, & \varepsilon \ge 0 \\ \ge, & \varepsilon \le 0 \end{cases}$$

From (A35) and (A36) we conclude that, given $q_1 = \bar{q}$, $q_2 = \bar{\bar{q}}$ is the larger root of (17) and (18). Furthermore, it is readily verified that for δ sufficiently large, there is a unique positive $\bar{q}$ for which the larger roots of (17) and (18) are the same and that $\bar{q} < \bar{\bar{q}}$.

Because from Lemma A10, $p_i^{**} = 1$, we have $R_i(0) \geq \bar{q}$. Because $\bar{q} > q^m$ and $R_0(\bar{q}) = 0$, $R_i(0) = \bar{q}$. Because, from Lemma (A10) $p_0^* = p_0^{**} = 0$, $R_0(i) = 0$.

From (A17), Lemma (A9) and because $p_j^{**} < 1$ (see the passage after (A17)), $R_0(\bar{q}) = 0$, and $R_i(0) = \bar{q}$, we have

(A37) $\quad V(i,i) = (1-p_j^{**})\delta V(0,0).$

If $p_j^{**} > 0$,

(A38) $\quad V(i,i) = (1-p_j^{**})\pi_i(\bar{q},0)\,\frac{1}{1-\delta} + p_j^{**}(\pi_i(\bar{q},\bar{q}) + \delta^2 V(i,i))$

From Lemma (A10) and (A11)

(A39) $\quad V(0,0) = \delta(1-p^{**})V(0,0)$ if $1 - p^{**} > 0$.

From Lemma (A10) and equation (A12)

(A40) $\quad V(0,0) = (1-p^{**})\left(\frac{\bar{q}(1-\bar{q})-f}{1-\delta} - e\right) + p^{**}(\pi_0(\bar{q},\bar{q}) + \delta^2 V(i,i)),$

if $p^{**} > 0$. Equation (A37) implies

(A41) $\quad V(i,i) \leq V(0,0).$

If $p^{**} = 1$, (A40) implies that $V(i,i) < V(0,0)$, a contradiction. Therefore $p^{**} < 1$, and from (A39)

(A42) $\quad V(0,0) = 0.$

If $p^{**} = 0$, then starting from state (0,0), a firm can play $\bar{q}$, which leads to positive profit

$$\frac{\bar{q}(1-\bar{q})}{1-\delta} - e,$$

a contradiction of (A42). Therefore $p^{**} > 0$, and, taking $\alpha_0 = p^{**}$, we conclude that

$$R_0(0) = \begin{cases} \bar{q}, \text{ with probability } \alpha_0 \\ \\ 0, \text{ with probability } 1 - \alpha_0 \end{cases}$$

where α_0 solves (19).

From (A41) and (A42), $V(i,i) = 0$. If $p_j^{**} = 0$, then, starting from (i,i), a firm can play $\bar{q}$ which leads to positive profit

$$\frac{\bar{q}(1-\bar{q})}{1-\delta}.$$

Therefore $p_j^{**} > 0$, and, taking $\alpha_i = p_j^{**}$, we conclude that

$$R_i(i) = \begin{cases} \bar{q} \text{ with probability } \alpha_i \\ \\ 0 \text{ with probability } 1 - \alpha_i, \end{cases}$$

where α_i solves (14).

We have thus demonstrated that symmetric equilibrium, if it exists, is unique. To confirm existence is a straightforward matter of checking that for sufficiently high δ, the dynamic reaction functions defined in the proposition form an equilibrium.

Q.E.D.

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